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Potential equivalence transformations for nonlinear diffusion–convection equations

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Abstract

Potential equivalence transformations (PETs) are effectively applied to a class of nonlinear diffusion–convection equations. For this class, all possible potential symmetries are classified and a theorem on their connection with point symmetries via PETs is also proved. It is shown that the known nonlocal transformations between equations under consideration are nothing but PETs. The action of PETs on sets of exact solutions of a fast diffusion equation is investigated.

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1. Introduction

In this paper, we consider a class of nonlinear diffusion–convection equations of the form

$$u_t = (d(u)u_x)_x + k(u)u_x \tag{1}$$

that have a number of applications in mathematical physics (see for instance [2, 3, 18, 25]). Equation 1 is also called Richard's equation [29, 36].

Here, $d = d(u)$ and $k = k(u)$ are arbitrary smooth functions of u, $d(u) \neq 0$. The linear case of (1) ($d, k = \text{const}$) was studied by Lie [20] in his classification of linear second-order PDEs with two independent variables. (See also a modern treatment of this subject in [23].)

Various classes of quasi-linear evolutionary equations in two independent variables that intersect class (1) were investigated by means of symmetry methods in [4, 9, 11, 19, 22, 24, 33, 36, 37]. The complete and strong group classification of (1) as well as a review of previous results on this subject were presented in [27].

To study nonlocal symmetries of PDEs in the framework of the local approach, Bluman *et al* [7, 8] proposed the notion of *potential symmetries*. A system of PDEs may admit symmetries of this kind when some of the equations can be written in a conserved form. After introducing potentials for PDEs written in the conserved form as additional dependent variables, we obtain a new (potential) system of PDEs. Any local invariance transformation of the obtained system induces a symmetry of the initial system. If transformations of some of the 'non-potential' variables explicitly depend on potentials, this symmetry is a nonlocal (potential) symmetry of the initial system. More details about potential symmetries and their applications can be found in [5, 7, 8]. Potential symmetries of (1) and its generalizations were studied by Sophocleous [30–32]. Other approaches for investigation of nonlocal symmetries of (1) were used in [1, 28]. Lisle [21] obtained a number of results concerning equivalence transformations of (1) and (4) and group classification in these classes. Unfortunately, these results, including the notion of potential equivalence transformations (PETs), were little known until now and were rediscovered by other scientists.

In this paper, we study in detail connections between symmetries and equivalence transformations of equations (1), corresponding potential systems and equations for the potential. For class (1), we prove a theorem on connection of the potential symmetries with local ones via PETs. It is shown that all the known nonlocal transformations between equations from the class under consideration are, in fact, PETs. In particular, they include the well-known transformations linearizing the u^{-2}-diffusion (named also Fujita–Storm) equation [6, 34], Fokas–Yortsos [12, 35] and Burgers equations [10, 13, 17] as well as the less known transformation of 'logarithmic nonlinearity' to 'power nonlinearity'. For some equations PETs are nonlocal symmetries. In such cases they generate additional equivalences on the corresponding sets of solutions and can be used, e.g., to construct new exact solutions from known ones.

Our paper is organized as follows. In section 2, known results on classical group analysis of diffusion–convection equations are adduced in a form which is suitable for purposes of our investigations. After formulating the statement of problem on classification of potential symmetries rigorously, we completely classify symmetries of such kind for the equations under consideration in section 3. Analysis of connections between potential and Lie symmetries of equations from class (1), which is given in section 4, is essentially based on the accuracy of the above classification results. The main theorem on reducibility of potential symmetries to point symmetries with potential and additional point equivalence transformations is proved for class (1). Section 5 is devoted to the demonstration of using PETs as nonlocal invariance transformations. As an example, the fast diffusion equation $u_t = (\ln u)_{xx}$ is considered in such framework.

2. Group classification of diffusion–convection equations

The exhaustive result on classical group classification of class (1) is presented by the statements adduced below [27].

Using the direct method, we construct the complete equivalence group including both continuous and discrete point transformations.

Theorem 1. *Any transformation from the equivalence group $G^{\sim}$ has the form*

$$\tilde{t} = \varepsilon_4 t + \varepsilon_1, \qquad \tilde{x} = \varepsilon_5 x + \varepsilon_7 t + \varepsilon_2, \qquad \tilde{u} = \varepsilon_6 u + \varepsilon_3,$$
$$\tilde{d} = \varepsilon_4^{-1}\varepsilon_5^2 d, \qquad \tilde{k} = \varepsilon_4^{-1}\varepsilon_5 k - \varepsilon_7,$$

where $\varepsilon_1, \ldots, \varepsilon_7$ are arbitrary constants, $\varepsilon_4\varepsilon_5\varepsilon_6 \neq 0$.

Corollary 1. *If the equations from class (1) are rewritten in the explicit conserved form*

$$u_t = (d(u)u_x - K(u))_x \tag{2}$$

Table 1. Results of group classification for class (1).

N	$d(u)$	$k(u)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_t, ∂_x
1	$\forall$	0	$\partial_t, \partial_x, 2t\partial_t + x\partial_x$
2	$e^{\mu u}$	e^u	$\partial_t, \partial_x, (\mu-2)t\partial_t + (\mu-1)x\partial_x + \partial_u$
3	e^u	u	$\partial_t, \partial_x, t\partial_t + (x-t)\partial_x + \partial_u$
4	e^u	0	$\partial_t, \partial_x, 2t\partial_t + x\partial_x, t\partial_t - \partial_u$
5	u^μ	u^ν	$\partial_t, \partial_x, (\mu-2\nu)t\partial_t + (\mu-\nu)x\partial_x + u\partial_u$
6	u^μ	$\ln u$	$\partial_t, \partial_x, \mu t\partial_t + (\mu x - t)\partial_x + u\partial_u$
7a	u^μ	0	$\partial_t, \partial_x, 2t\partial_t + x\partial_x, \mu t\partial_t - u\partial_u$
7b	u^{-2}	u^{-2}	$\partial_t, \partial_x, 2t\partial_t + u\partial_u, e^{-x}(\partial_x + u\partial_u)$
8	$u^{-4/3}$	0	$\partial_t, \partial_x, 2t\partial_t + x\partial_x, 4t\partial_t + 3u\partial_u, x^2\partial_x - 3xu\partial_u$
9	1	u	$\partial_t, \partial_x, t\partial_x - \partial_u, 2t\partial_t + x\partial_x - u\partial_u, t^2\partial_t + tx\partial_x - (tu+x)\partial_u$
10	1	0	$\partial_t, \partial_x, 2t\partial_t + x\partial_x, 2t\partial_x - xu\partial_u, 4t^2\partial_t + 4tx\partial_x - (x^2+t)u\partial_u, u\partial_u, h\partial_u$

where $K_u = -k$, then the corresponding equivalence group $\widehat{G}^{\sim}$ consists of the transformations

$$\tilde{t} = \varepsilon_4 t + \varepsilon_1, \qquad \tilde{x} = \varepsilon_5 x + \varepsilon_7 t + \varepsilon_2, \qquad \tilde{u} = \varepsilon_6 u + \varepsilon_3,$$
$$\tilde{d} = \varepsilon_4^{-1}\varepsilon_5^2 d, \qquad \tilde{K} = \varepsilon_4^{-1}\varepsilon_5\varepsilon_6 K + \varepsilon_7 u + \varepsilon_8,$$

where $\varepsilon_1, \ldots, \varepsilon_8$ are arbitrary constants, $\varepsilon_4\varepsilon_5\varepsilon_6 \neq 0$.

Theorem 2. *The Lie algebra of the kernel of principal groups of 1 is $A^{\ker} = \langle\partial_t, \partial_x\rangle$. A complete set of $G^{\sim}$-inequivalent equations (1) with the maximal Lie invariance algebra $A^{\max}$ not equal to $A^{\ker}$ is exhausted by cases given in table 1.*

Note 1. In table 1, $\mu, \nu =$ const. $(\mu, \nu) \neq (-2, -2), (0, 1)$ and $\nu \neq 0$ for Case 1.5. $\mu \neq -4/3, 0$ for Case 1.7a. The function $h = h(t, x)$ is an arbitrary solution of the linear heat equation ($h_t = h_{xx}$). Case 1.7b can be reduced to 1.7a ($\mu = -2$) by means of the additional equivalence transformation

$$\tilde{t} = t, \qquad \tilde{x} = e^x, \qquad \tilde{u} = e^{-x}u. \tag{3}$$

Note 2. Hereafter for convenience we use double numeration $T.N$ of classification cases, where T denotes the number of the table and N denotes the number of the respective case in table T. The notion 'equation $1.N$' ('system $2.N$') is used for the equation of form (1) (the system of form (4)) where the parameter functions take values from the corresponding case.

Note 3. The exponential Cases 1.2–1.4 can be regarded as limits of the power Cases 1.5–1.7a. More exactly,

$$\tilde{u} = 1 + \nu^{-1}u, \mu = \mu'\nu: \quad 1.5_{\mu,\nu} \to 1.2_{\mu'}, \nu \to +\infty,$$
$$\tilde{u} = 1 + \mu^{-1}u, \tilde{t} = \mu^2 t, \tilde{x} = \mu x: \quad 1.6_\mu \to 1.3, \mu \to +\infty,$$
$$\tilde{u} = 1 + \mu^{-1}u: \quad 1.7a_\mu \to 1.4, \mu \to +\infty.$$

The above limits are extended to the structure of the Lie invariance algebras and can be used to obtain exact solutions for the exponential cases from those for the power cases. Some partial cases of the above limits for diffusion equations ($k = 0$) were adduced in [7, 8].

3. Classification of potential symmetries

After rewriting equation (1) in the conserved form $u_t = (du_x - K)_x$ where $K' = -k$ and introducing the new (potential) unknown function $v = v(t,x)$, we obtain the equivalent system of PDEs (called the *potential* one)

$$v_x = u, \qquad v_t = du_x - K. \tag{4}$$

It follows from system (4) that the function v satisfies the equation

$$v_t = d(v_x)v_{xx} - K(v_x) \tag{5}$$

that is called the *potential equation* corresponding to equation (1). System (4) can be regarded as a Lie–Bäcklund transformation between equations (1) and (5).

Lisle proved in [21] that the Lie algebra of the equivalence group $G^{\sim}_{\text{pot}}$ for the class of systems (4) is

$$A^{\sim}_{\text{pot}} = \langle \partial_t, \partial_x, \partial_u + x\partial_v, \partial_v, t\partial_t - d\partial_d - K\partial_K, x\partial_x + v\partial_v + 2d\partial_d + K\partial_K,$$
$$u\partial_u + v\partial_v + K\partial_K, t\partial_x + u\partial_K, t\partial_v - \partial_K, v\partial_x - u^2\partial_u + 2ud\partial_d - uK\partial_K\rangle.$$

He also constructed the connected component of unity in $G^{\sim}_{\text{pot}}$ and attached some discrete equivalence transformations to it. We prove using the direct method that the transformation group obtained in such way coincides with the whole equivalence group $G^{\sim}_{\text{pot}}$.

Theorem 3. *Any transformation from $G^{\sim}_{\text{pot}}$ has the form*

$$\tilde t = \varepsilon_1 t + \varepsilon_2, \qquad \tilde x = \varepsilon'_1 x + \varepsilon'_2 v + \varepsilon'_3 t + \varepsilon'_4,$$
$$\tilde v = \varepsilon''_1 x + \varepsilon''_2 v + \varepsilon''_3 t + \varepsilon''_4, \qquad \tilde u = \frac{\varepsilon''_1 + \varepsilon''_2 u}{\varepsilon'_1 + \varepsilon'_2 u},$$
$$\tilde d = \frac{(\varepsilon'_1 + \varepsilon'_2 u)^2}{\varepsilon_1} d, \qquad \tilde K = \frac{\varepsilon'_1\varepsilon''_2 - \varepsilon'_2\varepsilon''_1}{\varepsilon'_1 + \varepsilon'_2 u}\frac{K}{\varepsilon_1}$$
$$-\frac{\varepsilon''_3}{\varepsilon_1} + \frac{\varepsilon'_3}{\varepsilon_1}\frac{\varepsilon''_1 + \varepsilon''_2 u}{\varepsilon'_1 + \varepsilon'_2 u},$$

where $\varepsilon_1, \varepsilon_2, \varepsilon'_i, \varepsilon''_i$ $(i = \overline{1,4})$ are arbitrary constants, $\varepsilon_1(\varepsilon'_1\varepsilon''_2 - \varepsilon'_2\varepsilon''_1) \neq 0$.

Definition 1. *We call the transformations from $G^{\sim}_{\text{pot}}$ potential equivalence transformations (PETs) for class 1 (or 2).*

Theorem 4. *The set $G^{\sim}_{\text{triv.pot}}$ of potential equivalence transformations which act on the arbitrary elements d and K trivially modulo $\widehat{G}^{\sim}$ is a normal subgroup of $G^{\sim}_{\text{pot}}$. The corresponding factor group can be identified with the group formed by the transformations*

$$\tilde t = t, \qquad \tilde x = x + \varepsilon v, \qquad \tilde u = \frac{u}{1+\varepsilon u},$$
$$\tilde v = v, \qquad \tilde d = (1+\varepsilon u)^2 d, \qquad \tilde K = \frac{K}{1+\varepsilon u}, \tag{6}$$

where ε is an arbitrary real, and the hodograph transformation of variables x and v

$$\tilde t = t, \qquad \tilde x = v, \qquad \tilde u = u^{-1},$$
$$\tilde v = x, \qquad \tilde d = u^2 d, \qquad \tilde K = -u^{-1}K. \tag{7}$$

Table 2. Results of group classification for systems (4) with respect to $G^{\sim}_{\text{triv. pot}}$-equivalence.

N	$d(u)$	$K(u)$	Basis of $A^{\max}$
0*	$\forall$	$\forall$	$\partial_t,\ \partial_x,\ \partial_v$
1*	$\forall$	0	$\partial_t,\ \partial_x,\ \partial_v,\ 2t\partial_t + x\partial_x + v\partial_v$
2*	$e^{\mu u}$	e^u	$\partial_t,\ \partial_x,\ \partial_v,\ (\mu-2)t\partial_t + (\mu-1)x\partial_x + \partial_u + ((\mu-1)v + x)\partial_v$
3*	e^u	u^2	$\partial_t,\ \partial_x,\ \partial_v,\ t\partial_t + (x+2t)\partial_x + \partial_u + (x+v)\partial_v$
4*	e^u	0	$\partial_t,\ \partial_x,\ \partial_v,\ 2t\partial_t + x\partial_x + v\partial_v,\ t\partial_t - \partial_u - x\partial_v$
5*	u^μ	$u^{\nu+1}$	$\partial_t,\ \partial_x,\ \partial_v,\ (\mu-2\nu)t\partial_t + (\mu-\nu)x\partial_x + u\partial_u + (\mu-\nu+1)v\partial_v$
6*	u^μ	$\ln u$	$\partial_t,\ \partial_x,\ \partial_v,\ (\mu+2)t\partial_t + (\mu+1)x\partial_x + u\partial_u + ((\mu+2)v - t)\partial_v$
7*	u^μ	$u\ln u$	$\partial_t,\ \partial_x,\ \partial_v,\ \mu t\partial_t + (\mu x + t)\partial_x + u\partial_u + (\mu+1)v\partial_v$
8*	u^μ	0	$\partial_t,\ \partial_x,\ \partial_v,\ 2t\partial_t + x\partial_x + v\partial_v,\ \mu t\partial_t - u\partial_u - v\partial_v$
1	$u^{-2}e^{\mu/u}$	$u\,e^{1/u}$	$\partial_t,\ \partial_x,\ \partial_v,\ (\mu-2)t\partial_t + ((\mu-1)x + v)\partial_x - u^2\partial_u + (\mu-1)v\partial_v$
2	$u^{-2}e^{1/u}$	u^{-1}	$\partial_t,\ \partial_x,\ \partial_v,\ t\partial_t + (x+v)\partial_x - u^2\partial_u + (v-2t)\partial_v$
3	$u^{-2}e^{1/u}$	0	$\partial_t,\ \partial_x,\ \partial_v,\ 2t\partial_t + x\partial_x + v\partial_v,\ t\partial_t - v\partial_x + u^2\partial_u$
4	$\frac{u^\mu}{(u+1)^{\mu+2}}$	$\frac{u^{\nu+1}}{(u+1)^\nu}$	$\partial_t,\ \partial_x,\ \partial_v,\ (\mu-2\nu)t\partial_t + ((\mu-\nu)x - v)\partial_x + u(u+1)\partial_u + (\mu-\nu+1)v\partial_v$
5	$\frac{u^\mu}{(u+1)^{\mu+2}}$	$u\ln\frac{u}{u+1}$	$\partial_t,\ \partial_x,\ \partial_v,\ \mu t\partial_t + (\mu x + v - t)\partial_x + u(u+1)\partial_u + (\mu+1)v\partial_v$
6	$\frac{u^\mu}{(u+1)^{\mu+2}}$	0	$\partial_t,\ \partial_x,\ \partial_v,\ 2t\partial_t + x\partial_x + v\partial_v,\ \mu t\partial_t + v\partial_x - u(u+1)\partial_u - v\partial_v$
7	$\frac{e^{\mu\arctan u}}{u^2+1}$	$\sqrt{u^2+1}\,e^{\nu\arctan u}$	$\partial_t,\ \partial_x,\ \partial_v,\ (\mu-2\nu)t\partial_t + ((\mu-\nu)x - v)\partial_x + (u^2+1)\partial_u +$ $(x + (\mu-\nu)v)\partial_v$
8	$\frac{e^{\mu\arctan u}}{u^2+1}$	0	$\partial_t,\ \partial_x,\ \partial_v,\ 2t\partial_t + x\partial_x + v\partial_v,\ \mu t\partial_t + v\partial_x - (u^2+1)\partial_u - x\partial_v$
9	u^{-2}	0	$\partial_t,\ \partial_v,\ 2t\partial_t + u\partial_u + v\partial_v,\ -vx\partial_x + u(ux+v)\partial_u + 2t\partial_v,$ $4t^2\partial_t - (v^2+2t)x\partial_x + u(v^2+6t+2xuv)\partial_u + 4tv\partial_v,$ $x\partial_x - u\partial_u,\ \phi\partial_x - \phi_v u^2\partial_u$
10	u^{-2}	u^{-1}	$\partial_t,\ \partial_v,\ 2t\partial_t + u\partial_u + v\partial_v,\ -v\partial_x + u^2\partial_u + 2t\partial_v,$ $4t^2\partial_t - (v^2+2t)\partial_x + 2u(uv+2t)\partial_u + 4tv\partial_v,$ $\partial_x,\ e^{-x}\phi\partial_x + e^{-x}(\phi - u\phi_v)u\partial_u$
11	1	$-u^2$	$\partial_t,\ \partial_x,\ 2t\partial_t + x\partial_x - u\partial_u,\ 2t\partial_x - \partial_u - x\partial_v,$ $4t^2\partial_t + 4tx\partial_x - 2(x+2ut)\partial_u - (x^2+2t)\partial_v,$ $\partial_v,\ e^{-v}(h_x - hu)\partial_u + e^{-v}h\partial_v$
12	1	0	$\partial_t,\ \partial_x,\ 2t\partial_t + x\partial_x - u\partial_u,\ 2t\partial_x - (xu+v)\partial_u - xv\partial_v,$ $4t^2\partial_t + 4tx\partial_x - ((x^2+6t)u + 2xv)\partial_u - (x^2+2t)v\partial_v,$ $u\partial_u + v\partial_v,\ h_x\partial_u + h\partial_v$

Here $\mu, \nu = \text{const.}$ $(\mu, \nu) \neq (-2, -2), (0, 1)$ and $\nu \neq -1, 0$ for Cases 2.5* and 2.4. $\mu \neq -2, 0$ for Cases 2.8* and 2.6. The functions $\phi = \phi(t, v)$ and $h = h(t, x)$ are arbitrary solutions of the linear heat equation ($\phi_t = \phi_{vv}$; $h_t = h_{xx}$).

Definition 2. *We will call (6) and (7) purely potential equivalence transformations for the class of PDEs (1).*

Studying potential symmetries of (1) is equivalent to solving the group classification problem in the class of systems (4) with respect to the (incomplete) equivalence group $G^{\sim}_{\text{triv. pot}}$. Let us note that potential symmetries of (1) were investigated in [30]. Using transformations from $G^{\sim}_{\text{triv. pot}}$, we essentially simplify, order and complete these results.

Theorem 5. *The Lie algebra of the kernel of principal groups of (4) is* $A^{\ker}_{\text{pot}} = \langle\partial_t, \partial_x, \partial_v\rangle$ $\left(=A^{\max}_{2.0}\right)$. *A complete set of* $G^{\sim}_{\text{triv. pot}}$*-inequivalent systems (4) with the maximal Lie invariance algebra* $A^{\max}$ *not equal to* $A^{\ker}_{\text{pot}}$ *is exhausted by cases given in table 2.*

To test some results presented in table 2, we used the unique program LIE by Head [16].

4. Analysis of classification results

Let us analyse the connections between cases from tables 1 and 2.

Cases 2.0*–2.8* completely correspond to Cases 1.0–1.7a: 2.0* $\leftrightarrow$ 1.0, 2.1* $\leftrightarrow$ 1.1, 2.2* $\leftrightarrow$ 1.2, 2.3* $\leftrightarrow$ 1.3, 2.4* $\leftrightarrow$ 1.4, 2.5* $\leftrightarrow$ $1.5_{\mu\neq-1}$, 2.6* $\leftrightarrow$ $1.5_{\mu=-1}$, 2.7* $\leftrightarrow$ 1.6, $2.8^*_{\mu\neq-4/3} \leftrightarrow 1.7\mathrm{a}_{\mu\neq-2}$. The constant multiplier in K ($\sim k$) can be change using equivalence transformations of the form $\tilde t = \varepsilon^2 t, \tilde x = \varepsilon x, \tilde u = u, \tilde v = \varepsilon v, \tilde d = d, \tilde K = \varepsilon K$ ($\sim \tilde k = \varepsilon k$), which nontrivially act only on the latter basis operators in Cases 2.3* and 1.3. The correspondence 2.7* $\to$ 1.6 are established with the equivalence transformation $\tilde t = t, \tilde x = -x, \tilde u = u, \tilde v = -v, \tilde d = d, \tilde K = -K + 1$. All the above correspondences also mean isomorphisms of $A^{\max}_{2.*}\big/\langle\partial_v\rangle$ and $A^{\max}_1$, which are realized by means of the projection to the space of (t, x, u) ($\to$) or the prolongation on the variable v ($\leftarrow$). Therefore, equation (1) has no pure potential symmetries for these values of d and k.

Cases $2.8^*_{\mu=-4/3}$ and 1.8 do not correspond to each other completely because the basis operator $x^2\partial_x - 3xu\partial_u$ from $A^{\max}_{1.8}$ cannot be prolonged onto v in a local manner and the algebra $A^{\max}_{2.8^*_{\mu=-4/3}}\big/\langle\partial_v\rangle$ is isomorphic to a proper subalgebra of $A^{\max}_{1.8}$.

There are pairs of 'starred' cases from table 2 which are equivalent with respect to the hodograph transformation (7): $2.5^*_{\mu,\nu} \leftrightarrow 2.5^*_{\mu',\nu'}$ ($\mu + \mu' = -2, \nu + \nu' = 1$), $2.6^*_{\mu} \leftrightarrow 2.7^*_{\mu'}$, $2.8^*_{\mu} \leftrightarrow 2.8^*_{\mu'}$ ($\mu + \mu' = -2$). (To exclude from consideration cases which are equivalent to other with respect to $G^{\sim}_{\mathrm{pot}}$, we have to assume additionally that, e.g., $\mu \geqslant -1$ and $\nu \geqslant \frac{1}{2}$ if $\mu = -1$ for Cases $2.5^*_{\mu,\nu}$ and 2.8^*_{μ}.) Therefore, the following statement is true.

Lemma 1. *Cases in the pairs* $(1.5_{\mu,\nu}, 1.5_{\mu',\nu'})(\mu + \mu' = -2, \nu + \nu' = 1)$, $(1.5_{\nu=-1}, 1.6)$, $(1.7\mathrm{a}_{\mu}, 1.7\mathrm{a}_{\mu'})$ $(\mu + \mu' = -2)$ *are equivalent with respect to PET (7).*

The algebras $A^{\max}_{2.1}$–$A^{\max}_{2.12}$ contain operators which are not projectible to the space (t, x, u), i.e., their coefficients corresponding to the variables t, x and u depend on v. Therefore, equation (1) for these values of d and K has purely potential symmetries.

Cases 2.1–2.6 including the corresponding Lie invariance algebras are reduced to 'starred' cases by means of using purely PETs (7) and (6): 2.1 $\to$ 2.2*, 2.2 $\to$ 2.3*, 2.3 $\to$ 2.4* (hodograph transformation (7)); 2.4 $\to$ 2.5*, 2.5 $\to$ 2.6*, 2.1 $\to$ 2.2* (transformation (6) with $\varepsilon = 1$).

Using the equivalence transformation $\tilde t = t, \tilde x = x - t, \tilde u = u, \tilde v = v + 2t, d = d, \tilde K = K - u - 2$ from $G^{\sim}_{\mathrm{triv.\,pot}}$, one can reduce the function K in Case 2.4 ($\nu = -2$) to the simpler form $\tilde K = u^{-1}$. Then the last operator has the form $(\mu + 4)t\partial_t + ((\mu + 2)x - v)\partial_x + u(u + 1)\partial_u + ((\mu + 3)v + 2t)\partial_v$. A similar statement is true for $\nu = -3$.

Cases 2.7 and 2.8 are most specific in the sense of reducibility to cases from table 1. There exist no transformations over the real field that reduce these cases to a simpler form. After considering equation (1) and system (4) over the complex field we can reduce Cases 2.7/2.8 to Cases $2.5^*_{\mu',\nu'}/2.8^*_{\mu'}$ where $\mu' = -\mathrm{i}\mu/2 - 1, \nu' = -\mathrm{i}\nu/2 - 1/2$ using the partial case of transformation (6):

$$\tilde t = -4t, \qquad \tilde x = -2x + 2\mathrm{i}v, \qquad \tilde v = 2x + 2\mathrm{i}v,$$
$$\tilde u = \frac{u - \mathrm{i}}{u + \mathrm{i}}, \qquad \tilde d = (u + \mathrm{i})^2 d, \qquad \tilde K = \frac{K}{u + \mathrm{i}}.$$

Cases $2.7_{\mu,\nu}$ and $2.7_{\mu',\nu'}$ (2.8_{μ} and $2.8_{\mu'}$) are equivalent iff $\mu = -\mu'$ and $\nu = -\nu'$ ($\mu = -\mu'$). The equivalence is realized by means of the transformation of changing signs of x and u simultaneously. To exclude from consideration cases which are equivalent to other with respect to $G^{\sim}_{\mathrm{pot}}$, e.g., we have to assume additionally $\mu \geqslant 0$ and $\nu \geqslant 0$ if $\mu = 0$ ($\mu \geqslant 0$).

The u^{-2}-diffusion, Fokas–Yortsos, Burgers and linear heat equations (Cases $1.7\mathrm{a}_{\mu=-2}$, 1.7b, 1.9 and 1.10 correspondingly) essentially are distinguished by the group classification of equations (1) in different ways. After introducing the potential v and replacing equations (1) with systems (4) (Cases 2.9–2.12), distinction between these cases and the others becomes more explicit because $A^{\max}_{2.9}$–$A^{\max}_{2.12}$ are isomorphic infinite-dimensional algebras and, using PET (7) and additional equivalence transformation (3), we can transform these cases to each other (see also [21]):

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\begin{array}{c}\hat u_{\hat t} = (\hat u^{-2}\hat u_{\hat x})_{\hat x} + \hat u^{-2}\hat u_{\hat x}\\ \hat v_{\hat x} = \hat u,\ \hat v_{\hat t} = \hat u^{-2}\hat u_{\hat x} - \hat u^{-1}\end{array}} \arrow[r, <->, "{\hat t = \tilde t,\ \hat x = \tilde v,\ \hat u = \tilde u^{-1},\ \hat v = \tilde x}"] \arrow[d, <->, "{\check t = \hat t,\ \check x = \mathrm{e}^{\hat x},\ \check u = \mathrm{e}^{-\hat x}\hat u,\ \check v = \hat v}"'] & \boxed{\begin{array}{c}\tilde u_{\tilde t} = \tilde u_{\tilde x\tilde x} + 2\tilde u\tilde u_{\tilde x}\\ \tilde v_{\tilde x} = \tilde u,\ \tilde v_{\tilde t} = \tilde u_{\tilde x} + \tilde u^2\end{array}} \arrow[d, <->, "{\tilde u = \frac{v_x}{v},\ v_t = v_{xx} \Longleftarrow t = \tilde t,\ x = \tilde x,\ u = \tilde u\,\mathrm{e}^{\tilde v},\ v = \mathrm{e}^{\tilde v}}"'] \\
\boxed{\begin{array}{c}\check u_{\check t} = (\check u^{-2}\check u_{\check x})_{\check x}\\ \check v_{\check x} = \check u,\ \check v_{\check t} = \check u^{-2}\check u_{\check x}\end{array}} \arrow[r, <->, "{\check t = t,\ \check x = v,\ \check u = u^{-1},\ \check v = x}"] & \boxed{\begin{array}{c}u_t = u_{xx}\\ v_x = u,\ v_t = u_x\end{array}}
\end{tikzcd}
$$

Therefore, all equations (1) having infinite-dimensional algebras of potential symmetries are either linear or linearizable. As one can see, the well-known Cole–Hopf transformation can be obtained as a combination of the above transformations. Only for the linear heat equation the potential symmetry algebra factorized with $\langle\partial_v\rangle$ is isomorphic to the maximal Lie invariance algebra. The isomorphism is not established with simple projection to the space (t, x, u). It is possible because of linearity and coincidence of the initial and potential equations.

The above analysis results in the following theorem.

Theorem 6. *All the symmetries presented in table 2 can be obtained from Lie symmetries of (1) by means of prolongation to the potential v and application of PETs (6) and (7) (over the complex field in Cases 2.7 and 2.8) and additional equivalence transformation (3) prolonged to v ($\tilde v = v$).*

Symmetry properties of systems (4) are connected in a more direct way with those of (5) than with those of (1) because systems (4) are simply the first prolongation [23] of (5) with respect to the variable x. Using this connection, we can easily solve the problem of group classification in the class of equations (5).

Theorem 7. *The equivalence group $\tilde G^{\sim}_{\mathrm{pot}}$ of the class of equations (5) and its Lie algebra $\tilde A^{\sim}_{\mathrm{pot}}$ are projections of $G^{\sim}_{\mathrm{pot}}$ and $A^{\sim}_{\mathrm{pot}}$ to the space (t, x, v). The Lie algebra of the kernel of principal groups of (5) is $\tilde A^{\mathrm{ker}}_{\mathrm{pot}} = \langle\partial_t, \partial_x, \partial_v\rangle$. A complete set of $\tilde G^{\sim}_{\mathrm{pot}}$-inequivalent equations (5) with the maximal Lie invariance algebra $A^{\max}$ not equal to $\tilde A^{\mathrm{ker}}_{\mathrm{pot}}$ is exhausted by Cases 2.0*–2.4*, $2.5^*_{\mu,\nu}$ ($\mu \geqslant -1$ and $\nu \geqslant \frac{1}{2}$ if $\mu = -1$), 2.6^*_μ, 2.8^*_μ ($\mu \geqslant -1$), 2.7 ($\mu \geqslant 0$ and $\nu \geqslant 0$ if $\mu = 0$), 2.8 ($\mu \geqslant 0$) and 2.12.*

5. PETs as nonlocal symmetry transformations

There exist equations in class (1) that are invariant with respect to nontrivial transformations from $G^{\sim}_{\mathrm{pot}}$. Thus, equation (1) admits transformations either (6), $\varepsilon = 1$ or (7) iff

$$\text{either}\quad d = u^{-2}F^1(u^{-1}),\qquad K = uG^1(u^{-1})\quad\text{or}$$
$$d = u^{-1}F^2(\ln u),\qquad K = u^{1/2}G^2(\ln u),$$

where F^1 and G^1 are periodic functions with the period equal to 1 and F^2 (G^2) is an even (odd) function. Such nonlocal symmetry transformations generate additional (with respect to Lie symmetries) equivalences in sets of solutions.

Consider, in more detail, the fast diffusion equation

$$u_t = (u^{-1}u_x)_x. \tag{8}$$

It is invariant with respect to transformation (7) which is additional to the usual Lie symmetry group $G^{\max}$ of equation (8). Action of elements of $G^{\max}$ on the solutions is given by the formula [24]

$$\tilde u(t,x) = \varepsilon_3^{-1}\varepsilon_4^2 u(\varepsilon_3 t + \varepsilon_1, \varepsilon_4 x + \varepsilon_2),$$

where $\varepsilon_1, \ldots, \varepsilon_4$ are arbitrary constants, $\varepsilon_3\varepsilon_4 \neq 0$.

Lemma 2. *The set of Lie invariant solutions of equation (8) is closed under transformation (7).*

Proof. Transformation (7) generates an adjoint action $\mathcal{H}$ on the Lie symmetry algebra

$$A^{\max}_{\rm pot} = \langle \partial_t, \partial_x, \partial_v, \hat D^1 = x\partial_x - 2u\partial_u - v\partial_v, \hat D^2 = 2t\partial_t + x\partial_x + v\partial_v\rangle$$

of the corresponding potential system (Case $2.8^*_{\mu=-1}$), which is determined in the following way: $\mathcal{H}(\partial_t) = \partial_t, \mathcal{H}(\partial_x) = \partial_v, \mathcal{H}(\partial_v) = \partial_x, \mathcal{H}(\hat D^1) = -\hat D^1, \mathcal{H}(\hat D^2) = \hat D^2$. Elements of the Lie symmetry algebra

$$A^{\max} = \langle \partial_t, \partial_x, D^1 = x\partial_x - 2u\partial_u, D^2 = 2t\partial_t + x\partial_x\rangle$$

of equation (8) is prolonged with respect to v ambiguously up to a term proportional to ∂_v. Therefore, transformation (7) correctly generates also an adjoint action $\mathcal{H}'$ on the classes of elements from $A^{\max}$, which differ each from other with terms proportional to ∂_v.

In view of the above, up to translations with respect to v (or x) any invariant solution of (8) gives an invariant solution of the potential system, which is transformed by (7) to an invariant solution of the same system, and the latter solution can be projected to an invariant solution of (8). □

All invariant solutions constructed in closed forms earlier with the classical Lie method were collected, e.g., in [26]. A complete list of $G^{\max}$-inequivalent solutions of such type are exhausted by the following ones:

$$\text{(1) } u = \frac{1}{1+\varepsilon e^{x+t}};\quad \text{(2) } u = e^x;\quad \text{(3) } u = \frac{1}{x - t + \mu t e^{-x/t}};\quad \text{(4) } u = \frac{2t}{x^2+\varepsilon t^2};$$
$$\text{(5) } u = \frac{2t}{\cos^2 x};\quad \text{(6) } u = \frac{-2t}{\cosh^2 x};\quad \text{(7) } u = \frac{2t}{\sinh^2 x}. \tag{9}$$

Here ε and μ are arbitrary constants, $\varepsilon \in \{-1, 0, 1\} \bmod G^{\max}$. The arrows denote the possible transformations of solutions (9) to each other by means of (7) up to translations with respect to x:

$$\circlearrowright(1)_{\varepsilon=0};\quad (1)_{\varepsilon=1} \longleftrightarrow (1)_{\varepsilon=-1,x+t<0};\quad \circlearrowright(1)_{\varepsilon=-1,x+t>0};\quad (2) \longleftrightarrow (3)_{\mu=0,x>t};$$
$$\circlearrowright(4)_{\varepsilon=0};\quad (5) \longleftrightarrow (4)_{\varepsilon=4};\quad (6) \longleftrightarrow (4)_{\varepsilon=-4,|x|<2|t|};\quad (7) \longleftrightarrow (4)_{\varepsilon=-4,|x|>2|t|}.$$

The fifth connection was known earlier [14, 28]. If $\mu \neq 0$ solution (3) from list (9) is mapped by (7) to the solution

$$(8)\ u = t\vartheta(\omega) - t + \mu t\, \mathrm{e}^{-\vartheta(\omega)}, \qquad \omega = x - \ln|t|,$$

which is invariant with respect to the algebra $\langle t\partial_t + \partial_x + u\partial_u\rangle$. Here, ϑ is the function determined implicitly by the formula $\int(\vartheta - 1 + \mu\, \mathrm{e}^{-\vartheta})^{-1}\, \mathrm{d}\vartheta = \omega$.

To find exact solutions of equation (8), other methods can also be used. Thus, Gandarias [15] found the new exact non-Lie solutions with the nonclassical symmetry method. We adduce the list of these solutions up to $G^{\max}$-equivalence, completing it with similar ones:

$$\begin{array}{lll} (1)\ u = \dfrac{\cos t}{\sin x - \sin t}; & (2)\ u = \dfrac{\cosh t}{\sinh x - \sinh t}; & \\ (3)\ u = \dfrac{-\sinh t}{\cosh x + \cosh t}; & (4)\ u = \dfrac{\sinh t}{\cosh x - \cosh t}; & (5)\ u = \dfrac{\cos t}{\cosh x + \sin t}. \end{array} \tag{10}$$

Solutions (10) can be presented in the form of compositions of two simple waves which move with the same velocities in opposite directions:

$$\begin{array}{ll} (1)\ u = \cot(x - t) + \tan(x + t); & (2)\ u = \coth(x - t) - \tanh(x + t); \\ (3)\ u = \tanh(x - t) - \tanh(x + t); & (4)\ u = \coth(x + t) - \coth(x - t). \end{array}$$

(We simplify the above representations by a scale transformation. The fifth solution admits such representation over the complex field only.) Solutions (4) and (5) are not adduced in [15] in any form.

Up to translations with respect to x, transformation (7) acts on the set of solutions (10) in the following way:

$$(1)_{\sin x>\sin t} \longleftrightarrow (5); \qquad (1)_{\sin x<\sin t} \longleftrightarrow (5)|_{x\to -x};$$
$$\circlearrowright(2)_{x<t}; \qquad (2)_{x>t} \longleftrightarrow (2)_{x>t}|_{x\to -x}; \qquad (3) \longleftrightarrow (4)_{|x|<|t|}; \qquad \circlearrowright(4)_{|x|>|t|}.$$

The latter actions can be interpreted in terms of actions of transformation (7) on the nonclassical symmetry operators which correspond to solutions (10).

6. Conclusion

It was proved above that any nonlinear diffusion–convection equation having nontrivial potential symmetries can be reduced to another diffusion–convection equation with potential equivalence transformations such that all symmetries will become point. This result generates a number of questions, and each is an interesting problem. Are similar statements right for more general classes of differential equations? Does a differential equation exist, potential symmetries of which cannot be constructed from point symmetries of an equation equivalent to the initial one via potential transformations? Could the result be generalized to other kinds of symmetries (e.g., nonclassical, conditional and approximate ones)? We believe that solving the above problems will allow us to understand deeper the essence of potential symmetries.

We have also investigated the other potential forms of equations (1). These results will be published in our further paper on the subject.

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